Homework #2
CS170, Spring 2025

Due Monday, Feb 24, at Noon

General Instructions. Please start early, and allow at least 10 hours of work for this homework. Please keep in mind the late day policy on the course website.

Collaboration. Please carefully check the collaboration policy on the course website. When in doubt, ask an instructor.

Consulting outside sources. Please carefully check the policy regarding outside sources on the course website. Again, when in doubt, ask an instructor.

Submission. Homework submission will be via Gradescope — instructions can be found on Piazza. If you choose to type up your homework, we highly recommend using LaTeX (the text markup language we are also using for this assignment). You will likely need to learn LaTex eventually, so might as well start now. That being said, handwritten homework, or homework prepared using any word-processing software, is perfectly fine.

Problem 1. (10 points)
Prove the following statements:

(a) [2 points]. If an integer is not divisible by 2, then it cannot be divisible by 4. Generalize this statement to all integers and their multiples.

(b) [2 points]. If n is divisible by 3, then $n^2 + 3n$ is divisible by 18.

(c) [6 points]. If n is an integer, then $\frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$ is an integer.
Hint: What number must $x(x+1)(x+2)$ be divisible by, and why?

Problem 2. (10 points)

(a) [4 points]. Prove that an integer $n \geq 1$ is a perfect square (i.e., can be expressed as $n = k^2$ for some integer k) if and only if every power in its prime decomposition is even.

(b) [6 points]. Prove that, for any integer $n \geq 1$, its square root $\sqrt{n}$ is either an integer or is irrational.
Hint: Use part (a).

Problem 3. (10 points)
Use induction to prove the following statements:

(a) [5 points]. $6^n - 1$ is divisible by 5 for all positive integers n.

(b) [5 points]. $5^n + 3^n \geq 2^{2n+1}$ for all positive integers n.
Hint: Try to get your inductive step into the form $4(5^k + 3^k) + f(k)$, for some formula $f(k)$.

Problem 4. (10 points)
You visit a small country that only uses 3-cent and 7-cent coins. This system seems strange to you, but the locals insist that any quantity greater than or equal to 9 cents can be represented in their currency.

(a) (2 points). Prove that they are wrong.

(b) (8 points). The country introduces 11-cent coins. Prove that any quantity greater than or equal to 9 cents can be represented with 3-cent, 7-cent, and 11-cent coins.

Problem 5. (10 points)
Let a and r be real numbers such that $a \geq 0$ and $0 \leq r < 1$. Consider the sequence $a_0, a_1, a_2, \ldots$ defined as follows

- $a_0 = a$
- $a_i = r \times a_{i-1}$ for integers $i > 0$

Using induction, show that $\sum_{i=0}^{n} a_i \leq \frac{a}{1-r}$ for all integers $n \geq 0$.
Hint: $\frac{a(1-r^n)}{1-r}$

Problem 6. (10 points)
Consider a group of friends who use Instagram. You are told that for any two friends a and b in this group, either a follows b or b follows a (and possibly, but not necessarily, both). Show that there is at least one person c in this group such that every other person d in the group either follows c or follows someone who follows c.