Homework #3
CS170, Spring 2025

Due Friday, March 14, at Noon

General Instructions. Please start early, and allow at least 10 hours of work for this homework. Please keep in mind the late day policy on the course website. Please provide a rigorous mathematical proof for all your claims, and present runtime guarantees for your algorithms using asymptotic (big-$O/\Omega/\Theta$) notation, unless stated otherwise. You may assume that all basic arithmetic operations (multiplication, subtraction, division, comparison, etc.) take constant time.

Collaboration. Please carefully check the collaboration policy on the course website. When in doubt, ask an instructor.

Consulting outside sources. Please carefully check the policy regarding outside sources on the course website. Again, when in doubt, ask an instructor.

Submission. Homework submission will be via Gradescope — instructions can be found on Piazza. If you choose to type up your homework, we highly recommend using LaTeX (the text markup language we are also using for this assignment). You will likely need to learn LaTex eventually, so might as well start now. That being said, handwritten homework, or homework prepared using any word-processing software, is fine.

Problem 1. (15 points)
Compare the following runtimes using big-$\mathcal{O}$ and big-Θ. All logarithms below are base 2. You do not need to prove your claims.

- $n \log n$
- n^n
- $n^{\log n}$
- $2^{\log^2 n}$
- $n!$
- $\log(n!)$

Problem 2. (16 points)
Suppose you are given an array $a = [a_0, \ldots, a_{n-1}]$ of n numbers sorted in nondecreasing order, as well as specific number x to search for. You must find i such that $a_i = x$ if it exists, or else output "not found". There is a well-known algorithm for this, which you might be familiar with: binary search. Though binary search is typically implemented recursively, consider instead the following pseudocode for an iterative implementation of binary search.

BinarySearch(a, x):

- $n = \text{length}(a)$
- $l = 0$, $r = n - 1$.
- While $l \leq r$
  - $m = (l + r)/2$
  - If $a[m] = x$ then Return m
  - Else if $a[m] < x$ then $l = m + 1$
  - Else $r = m - 1$.
- Return "not found"

(a) [8 points]. Prove correctness for this implementation of binary search, using a loop invariant.

(b) [8 points]. Analyze the worst-case runtime of this algorithm using big-$\mathcal{O}$ notation. As part of your proof, you can use a loop invariant to measure progress of your algorithm.

Problem 3. (12 points)
Suppose you design two algorithms for the same problem. Both algorithms run in time $T(1) = 1$ on inputs of size 1. For larger inputs $n > 1$, the runtime of the first algorithm satisfies $T(n) = 3T(n/3) + n$, and the runtime of the second algorithm satisfies $T(n) = 5T(n/6) + 10n$. Which algorithm is asymptotically faster? Justify your claim using the tree method for solving recurrences. Your justification does not have to be completely formal, in that you can feel free to ignore issues having to rounding integers up or down; these issues are messy to deal with, but ultimately turn out to be inconsequential.

Problem 4. (12 points)
Given the following sequence, prove that $f(n)$ is the $(n+1)$st Fibonacci number, for all integers $n > 0$:

$$f(n) = 1 + \sum_{i=0}^{n-2} f(i) \text{ and } f(0) = 0 \text{ and } f(1) = 1$$

Problem 5. (12 points)
Let S be the set of integers defined inductively as follows.

- $7 \in S$
- If $n \in S$ then both $3n \in S$ and $n^2 \in S$.

(a) [6 points]. Show that every $n \in S$ can be expressed as $n = 14q + 7$ for some integer q.

(b) [6 points]. Give pseudocode for a recursive algorithm which determines whether a given number n is in S.

(c) [6 points BONUS]. Analyze the worst-case runtime of your algorithm as a function of n.

Problem 6. (24 points)
Prove or disprove the following statements involving sets A, B, and C.

(a) If $A \times C = B \times C$ then $A = B$.

(b) If $A \times C = B \times C$ and $C \neq \emptyset$ then $A = B$.

(c) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(d) If $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$ then $A = B = C$.

(e) $|A - B| = |A| - |B|$

(f) $|A \cap B| + |A \cup B| = |A| + |B|$

(g) $A - (B \cap C) = (A - B) \cup (A - C)$

(h) If $\mathcal{P}(A) = \mathcal{P}(B)$ then $A = B$, where $\mathcal{P}(.)$ denotes power-set.

Problem 7. (15 points)
For each of the following functions, determine if it is injective, surjective, both (i.e., bijective), or neither. Prove your claims.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

(b) $f : \mathbb{N} \to \mathbb{R}$, $f(x) = \sqrt{x}$

(c) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^+$, $f(x, y) = \frac{x}{y}$. Here, we are using the shorthand $A^+ = \{x \in A : x > 0\}$.

(d) $f : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{N})$, $f(A) = A \cap \mathbb{N}$.

(e) $f : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$, $f(A) = \mathbb{Z} - A$.