Homework #4
CS170, Spring 2025

Due Friday, April 11, by Noon

General Instructions. Please start early, and allow at least 10 hours of work for this homework. Please keep in mind the late day policy on the course website.

Collaboration. Please carefully check the collaboration policy on the course website. When in doubt, ask an instructor.

Consulting outside sources. Please carefully check the policy regarding outside sources on the course website. Again, when in doubt, ask an instructor.

Submission. Homework submission will be via Gradescope — instructions can be found on Piazza. If you choose to type up your homework, we highly recommend using LaTeX (the text markup language we are also using for this assignment). You will likely need to learn LaTex eventually, so might as well start now. That being said, handwritten homework, or homework prepared using any word-processing software, is fine.

Recommended practice problems (do not hand in): LS Exercises 5.1-5.3, 5.5, 5.8-5.10, 6.2, 6.5, 6.6, 6.7, 12.1-12.3. For even more practice, try to tackle as many problems as you can from Rosen (7th Edition) Chapters 2.1-2.4, 9.1, 9.2, 9.5, 9.6, 1.4-1.5, and 5.1-5.3.

Problem 1. (12 points)
Prove or disprove the following statements.

(a) If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.[1]

(b) A function $f : A \to A$ is injective if and only if it is surjective.

(c) If A is a finite set, then a function $f : A \to A$ is injective if and only if it is surjective.

Problem 2. (16 points)
For a function $f : \mathbb{N} \to \mathbb{N}$, we say it is *monotone non-decreasing* if $x \leq y \implies f(x) \leq f(y)$ for all $x, y \in \mathbb{N}$. We also say f is *unbounded* if for every $y \in \mathbb{N}$, there exists $x \in \mathbb{N}$ such that $f(x) \geq y$. Recall from class that the *range* of f is defined by $\text{range}(f) = \{f(x) : x \in \mathbb{N}\}$. Prove the following.

(a) [4 points]. f is unbounded if and only if its range has infinite cardinality.

(b) [12 points]. If f is monotone non-decreasing and its range has finite cardinality, then there is some $x \in \mathbb{N}$ with $f(x) = x$.

Problem 3. (12 points)
A group of six friends play some games of ping-pong with these results: Amy beats Elise, Elise beats Carl, Frank beats Elise, Amy beats Bob, Carl beats Dave, Bob beats Carl, Bob beats Dave, Frank beats Bob, Frank beats Amy. Consider the relation $R = \{(x, y) : x \text{ has beaten } y\}$.

(a) [4 points]. Draw the directed acyclic graph G representing R.

(b) [4 points]. Is R reflexive? Irreflexive? Symmetric? Asymmetric? Antisymmetric? Transitive? Total? A partial Order?

(c) [4 points]. The players want to rank themselves. Find every possible topological order of G.

Problem 4. (16 points)
A *preorder* is a single-set relation which is reflexive and transitive, but need not be antisymmetric. In other words, some different objects may be equated by a pre-order, unlike in a partial order. Notice that this makes preorders a generalization of both equivalence relations and partial orders. In this problem, we will see how a preorder can be thought of as just a partial order on equivalence classes defined in a natural way.

(a) [4 points]. As an example, consider the familiar asymptotic order relation on functions from $\mathbb{N}$ to $\mathbb{N}$: f is related to g if $f(n) = \mathcal{O}(g(n))$. Show that this is a preorder, but not a partial order or a total order.[2]

[1] Recall that $\mathcal{P}(A)$ denotes the power set of A.

[2] I recommend that you convince yourself (for your own education, without handing anything in) that the same is true for $\models$ on propositional formulae, and for reachability in directed graphs.

(b) [4 points]. Consider any preorder R on any universe U. Define the following relation $\sim_R$ on U as follows: $a \sim_R b$ if both $(a, b) \in R$ and $(b, a) \in R$. Show that $\sim_R$ is an equivalence relation.

(c) [2 points]. What would $\sim_R$ be if R were the relation from part (a)? (Hint: use a familiar symbol related to big-$\mathcal{O}$)

(d) [4 points]. Let $U/\sim_R$ denote the equivalence classes of $\sim_R$. (The forwards slash is often called the "quotient operator" in many areas of mathematics, and serves to group together "equivalent" elements in a universe). Define the following relation $\preceq_R$ on $U/\sim_R$: $A \preceq_R B$ if there is $a \in A$ and $b \in B$ such that $(a, b) \in R$. Show that $\preceq_R$ is a partial order.

(e) [2 points]. Would anything change in (d) if we instead defined $\preceq_R$ as follows: $A \preceq_R B$ if for all $a \in A$ and $b \in B$ we have $(a, b) \in R$?

Problem 5. (8 points)
Recall our definition of the big-Oh relation from class:

> For $f, g : \mathbb{N} \to \mathbb{N}$, we say $f = \mathcal{O}(g)$ if there exist constants n_0 and c such that $f(n) \leq cg(n)$ for all $n \geq n_0$.

Consider the following alternative definition.

> For $f, g : \mathbb{N} \to \mathbb{N}$, we say $f = \mathcal{O}(g)$ if there exists a constant c such that $f(n) \leq cg(n)$ for all $n \in \mathbb{N}$.

Show that the two definitions are equivalent — i.e., it doesn't matter which definition you use, as they result in the same relation on functions. You may use the definition of natural numbers which excludes 0, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$.[3]

Problem 6. (8 points)
For the following definitions:

- $P(x)$: x is a professor.
- $S(x)$: x is a student.
- $A(x, y)$: x annoys y.

Translate the following English sentences into first-order logic, for a universe of all people:

(a) All professors annoy all students.

(b) Some professor annoys all students.

(c) Only professors annoy students.

(d) Some student is annoyed by all professors

[3] This avoids some trivial cases which are not of interest to us when analyzing runtimes or most natural computational resources, which are typically non-zero.