Homework #5
CS170, Spring 2025

Due Friday, May 2, by 11:59pm

General Instructions. Please start early, and allow at least 10 hours of work for this homework. Please keep in mind the late day policy on the course website.

Collaboration. Please carefully check the collaboration policy on the course website. When in doubt, ask an instructor.

Consulting outside sources. Please carefully check the policy regarding outside sources on the course website. Again, when in doubt, ask an instructor.

Submission. Homework submission will be via Gradescope — instructions can be found on Piazza. If you choose to type up your homework, we highly recommend using LaTeX (the text markup language we are also using for this assignment). You will likely need to learn LaTex eventually, so might as well start now. That being said, handwritten homework, or homework prepared using any word-processing software, is fine.

Recommended practice problems (do not hand in): LS Exercises 12.5-12.7, 13.2,13.3, 13.5, 13.7, 13.8, 13.9, 16.3, 16.5, 17.8. For even more practice, try to tackle as many problems as you can from Rosen (7th Edition) Chapters 1.4-1.8, 10.1-10.3, 10.7, 11.1-11.3 (skip questions that involve notions we didn't mention in lecture).

Problem 1. (18 points)
Indicate whether the following statements are true or false, and briefly explain your answers. Note that $\equiv$ and $\models$ denote equivalence and entailment, respectively, in the semantic sense of logic.

(a) $\neg(\forall x A(x) \ \wedge \ \exists y B(y)) \equiv \neg\forall x A(x) \vee \neg\exists y B(y)$

(b) $A(a) \vee A(b) \vee A(c) \vee \ldots \vee A(y) \vee A(z) \equiv \exists x A(x)$

(c) $\neg(A(a) \vee A(b)) \equiv \neg A(a) \wedge \neg A(b)$

(d) $\forall x(p \implies Q(x)) \equiv p \implies \forall x\ Q(x)$

(e) $\exists x\exists y A(x,y) \equiv \exists y\exists x A(x,y)$

(f) $\forall x\forall y A(x,y) \equiv \forall y\forall x A(x,y)$

(g) $\forall x\exists y A(x,y) \equiv \forall x\exists y A(y,x)$

(h) $\forall z P(z,z) \models \forall x\exists y P(x,y)$

(i) $\forall x\exists y P(x,y) \models \forall z P(z,z)$

Problem 2. (12 points)
We used case analysis in some of our proofs in class, and Shaddin mentioned that this is justified by the rules of logic. In this problem, we will make this jusification more precise.

(a) [2 points]. Recall problem 6c from Hw1. Explain how this justifies a case analysis with two cases.

(b) [5 points]. Using induction, show that the following holds in propositional logic for any constant k.

$$\begin{array}{l} p_1 \implies r \\ p_2 \implies r \\ \ldots \\ p_k \implies r \\ \hline \therefore (p_1 \vee p_2 \vee \ldots \vee p_k) \implies r \end{array}$$

Explain how this justifies case analyses with any finite number of cases.

(c) [5 points]. Let $P(x)$ and R be formulas in first order logic, and suppose that the variable x does not appear in R. Show, using the rules of inference of first order logic presented in class, that $\forall x(P(x) \Rightarrow R)$ entails $(\exists x P(x)) \Rightarrow R$. Your (short) proof should use no more than four applications of the inference rules from class. Explain how this justifies a case analysis with any number of cases (possibly infinite).

Problem 3. (12 points)
Let G be a digraph with nodes $V = \{a, b, c, d, e\}$. For each of the following sets of edges E, describe how connected the graph would be (disconnected, unilaterally connected, weakly connected, or strongly connected), and whether it contains a cycle. Briefly explain your answers.

- $E = \{(a, b), (c, a), (c, b), (d, b), (b, e), (e, c), (e, d)\}$
- $E = \{(a, c), (b, c), (b, d), (c, d), (d, a), (d, e)\}$
- $E = \{(a, c), (b, d), (c, b), (d, c)\}$
- $E = \{(a, c), (b, d), (c, b), (e, c)\}$

Problem 4. (10 points)
Show In this problem, we will explore some properties of directed acyclic graphs (DAGs).

(a) [8 points] . Show that every DAG has a node with in-degree 0.

(b) [2 points] . By referencing part (a), but without replicating the proof, show that every DAG has a node with out-degree 0.

Problem 5. (12 points)
A tree is an undirected graph that is (a) connected, and (b) acyclic. You will find in your book a proof that (a) and (b) imply that (c) the number of edges is one fewer than the number of nodes. Show that (a) and (c) imply (b), then show that (b) and (c) imply (a). Conclude that any two of the properties (a), (b), (c) is an equally valid definition of a tree.

Problem 6. (6 points)
Consider a directed graph $G = (V, E)$, where each $e \in E$ is labeled with a weight (a.k.a. length or distance) $w_e \geq 0$. For $u, v \in V$, let $d(u, v)$ denote the (weighted) shortest path distance from u to v. Show that shortest path distances satisfy the triangle inequality: $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V$.

Problem 7. (10 points)
The n dimensional cube $Q_n = (V_n, E_n)$ is an undirected graph that can be inductively defined as follows:

- Q_0 is a single node with no edges
- For $n \geq 0$, Q_{n+1} consists of two copies of Q_n, plus an edge between each node and its copy. More precisely:
  - $V_{n+1} = \{(v, 0) : v \in V_n\} \cup \{(v, 1) : v \in V_n\}$
  - $E_{n+1} = \{((u, 0), (v, 0)) : (u, v) \in E_n\} \cup \{((u, 1), (v, 1)) : (u, v) \in E_n\} \cup \{((u, 0), (u, 1)) : u \in V_n\}$

Show using induction that Q_n has 2^n nodes and $n2^{n-1}$ edges.