

CS675: Convex and Combinatorial Optimization
Fall 2026
Duality of Convex Optimization Problems

Instructor: Shaddin Dughmi

Outline

1 The Lagrange Dual Problem

2 Duality

3 Optimality Conditions

Recall: Optimization Problem in Standard Form

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad \text{for } i = 1, \dots, m. \\ & h_i(x) = 0, \quad \text{for } i = 1, \dots, k.\end{array}$$

- For convex optimization problems in standard form, f_i is convex and h_i is affine.
- Let $\mathcal{D}$ denote the domain of all these functions (i.e. when their value is finite)

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This Lecture + Next

We will develop duality theory for convex optimization problems, generalizing linear programming duality.

Running Example: Linear Programming

We have already seen the standard form LP below

$$\begin{array}{ll}\text{maximize} & c^\top x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0\end{array}$$

$$\begin{array}{ll}-\text{minimize} & -c^\top x \\ \text{subject to} & Ax - b \preceq 0 \\ & -x \preceq 0\end{array}$$

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Along the way, we will recover the following standard form dual

$$\begin{array}{ll}\text{minimize} & y^\top b \\ \text{subject to} & A^\top y \succeq c \\ & y \succeq 0\end{array}$$

The Lagrangian

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad \text{for } i = 1, \dots, m. \\ & h_i(x) = 0, \quad \text{for } i = 1, \dots, k.\end{array}$$

Basic idea of Lagrangian duality is to relax/soften the constraints by replacing each with a linear “penalty term” or “cost” in the objective.

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The Lagrangian Function

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^k \nu_i h_i(x)$$

- λ_i is **Lagrange Multiplier** for i 'th inequality constraint
 - Required to be nonnegative
- ν_i is **Lagrange Multiplier** for i 'th equality constraint
 - Allowed to be of arbitrary sign

The Lagrange Dual Function

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad \text{for } i = 1, \dots, m. \\ & h_i(x) = 0, \quad \text{for } i = 1, \dots, k.\end{array}$$

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The Lagrange Dual Function

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^k \nu_i h_i(x) \right)$$

- Observe: g is a concave function of the Lagrange multipliers
- We will see: Its quite common for the Lagrange dual to be unbounded ($-\infty$) for some λ and ν
- By convention, domain of g is (λ, ν) s.t. $g(\lambda, \nu) > -\infty$

Langrange Dual of LP

$$\begin{array}{ll}\text{minimize} & -c^\top x \\ \text{subject to} & Ax - b \preceq 0 \\ & -x \preceq 0\end{array}$$

First, the Lagrangian function

$$\begin{aligned}L(x, \lambda) &= -c^\top x + \lambda_1^\top (Ax - b) - \lambda_2^\top x \\ &= (A^\top \lambda_1 - c - \lambda_2)^\top x - \lambda_1^\top b\end{aligned}$$

Langrange Dual of LP

$$\begin{array}{ll}\text{minimize} & -c^T x \\ \text{subject to} & Ax - b \preceq 0 \\ & -x \preceq 0\end{array}$$

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And the Lagrange Dual

$$\begin{aligned}g(\lambda) &= \inf_x L(x, \lambda) \\ &= \begin{cases} -\infty & \text{if } A^T \lambda_1 - c - \lambda_2 \neq 0 \\ -\lambda_1^T b & \text{if } A^T \lambda_1 - c - \lambda_2 = 0 \end{cases}\end{aligned}$$

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So we restrict the domain of g to λ satisfying $A^T \lambda_1 - c - \lambda_2 = 0$

Interpretation: “Soft” Lower Bound

$$\begin{array}{ll}\min & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad \text{for } i = 1, \dots, m. \\ & h_i(x) = 0, \quad \text{for } i = 1, \dots, k.\end{array}$$

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Fact

$g(\lambda, \nu)$ is a lowerbound on $\text{OPT}(\text{primal})$ for every $\lambda \succeq 0$ and $\nu \in \mathbb{R}^k$.

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Fact

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Proof

- Every primal feasible x incurs nonpositive penalty by $L(x, \lambda, \nu)$
- Therefore, $L(x^*, \lambda, \nu) \leq f_0(x^*)$
- So $g(\lambda, \nu) \leq f_0(x^*) = \text{OPT}(\text{Primal})$

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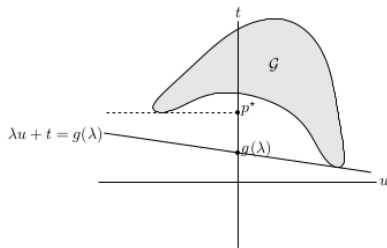
Interpretation

- A “hard” feasibility constraint can be thought of as imposing a penalty of $+\infty$ if violated, and a penalty/reward of 0 if satisfied
- Lagrangian imposes a “soft” linear penalty for violating a constraint, and a reward for slack
- Lagrange dual finds the optimal subject to these soft constraints

Interpretation: Geometric

Most easily visualized in the presence of a single inequality constraint

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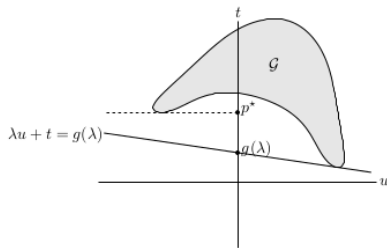


- Let $\mathcal{G}$ be attainable constraint/objective function value tuples
 - i.e. $(u, t) \in \mathcal{G}$ if there is an x such that $f_1(x) = u$ and $f_0(x) = t$
- $p^* = \inf \{t : (u, t) \in \mathcal{G}, u \leq 0\}$
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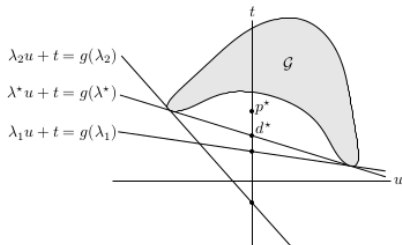


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 - $g(\lambda) = \inf \{\lambda u + t : (u, t) \in \mathcal{G}\}$
- $\lambda u + t = g(\lambda)$ is a supporting hyperplane to $\mathcal{G}$ pointing northeast
 - Must intersect vertical axis below p^*
 - Therefore $g(\lambda) \leq p^*$

The Lagrange Dual Problem

This is the problem of finding the best lower bound on $\text{OPT}(\text{primal})$ implied by the Lagrange dual function

$$\begin{array}{ll}\text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0\end{array}$$



- Note: this is a convex optimization problem, regardless of whether primal problem was convex
- By convention, sometimes we add “dual feasibility” constraints to impose “nontrivial” lowerbounds (i.e. $g(\lambda, \nu) \geq -\infty$)
- (λ^*, ν^*) solving the above are referred to as the **dual optimal** solution

Langrange Dual Problem of LP

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Recall

Our Lagrange dual function for the above minimization LP (to the right), defined over the domain $A^\top \lambda_1 - c - \lambda_2 = 0$.

$$g(\lambda) = -\lambda_1^\top b$$

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Another Example: Conic Optimization Problem

$$\begin{array}{ll}\text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \in K\end{array}$$

- $x \in K$ can equivalently be written as $z^\top x \leq 0, \forall z \in K^\circ$

$$\begin{aligned}L(x, \lambda, \nu) &= c^\top x + \nu^\top (Ax - b) + \sum_{z \in K^\circ} \lambda_z \cdot z^\top x \\ &= (c - A^\top \nu + \sum_{z \in K^\circ} \lambda_z \cdot z)^\top x + \nu^\top b\end{aligned}$$

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- Can think of $\lambda \succeq 0$ as choosing some $s \in K^\circ$

$$L(x, s, \nu) = (c - A^\top \nu + s)^\top x + \nu^\top b$$

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Weak Duality

Primal Problem

$$\min f_0(x)$$

s.t.

$$f_i(x) \leq 0, \quad \forall i = 1, \dots, m.$$

$$h_i(x) = 0, \quad \forall i = 1, \dots, k.$$

Dual Problem

$$\max g(\lambda, \nu)$$

s.t.

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Weak Duality

Primal Problem

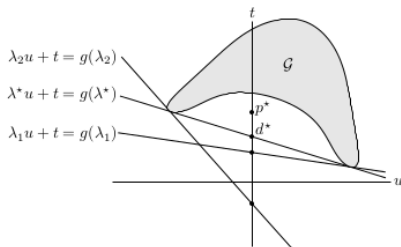
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Weak Duality

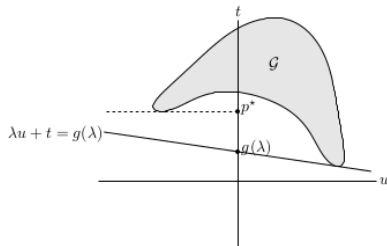
$$OPT(dual) \leq OPT(primal).$$



- We have already argued holds for every optimization problem
- **Duality Gap**: difference between optimal dual and primal values

Recall: Geometric Interpretation of Weak Duality

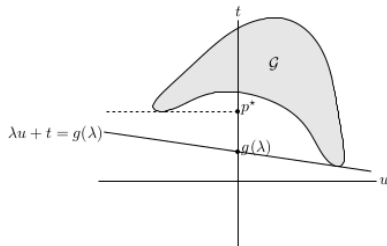
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Fact

The equation $\lambda u + t = g(\lambda)$ defines a supporting hyperplane to $\mathcal{G}$, intersecting t axis at $g(\lambda) \leq p^*$.

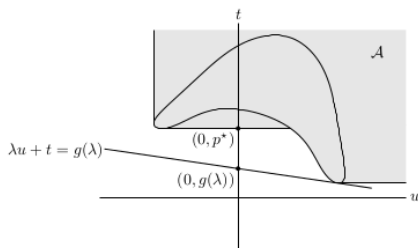
Strong Duality

We say strong duality holds if $OPT(dual) = OPT(primal)$.

- Equivalently: there exists a setting of Lagrange multipliers so that $g(\lambda, \nu)$ gives a tight lowerbound on primal optimal value.
- In general, does not hold for non-convex optimization problems
- Usually, but not always, holds for convex optimization problems.
 - Mild assumptions, such as **Slater's condition**, needed.

Geometric Proof of Strong Duality

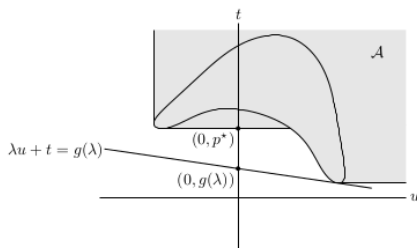
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- Let $\mathcal{A}$ be everything northeast (i.e. “worse”) than $\mathcal{G}$
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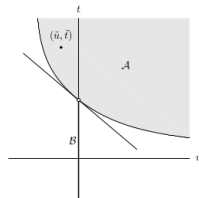
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Geometric Proof of Strong Duality

minimize $f_0(x)$
subject to $f_1(x) \leq 0$

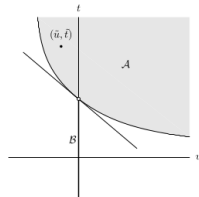


Fact

When f_0 and f_1 are convex, $\mathcal{A}$ is convex.

Geometric Proof of Strong Duality

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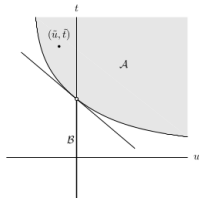
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Proof

- Assume (u, t) and (u', t') are in $\mathcal{A}$

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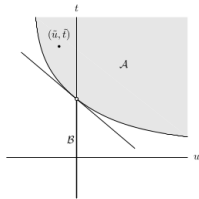
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Proof

- Assume (u, t) and (u', t') are in $\mathcal{A}$
- $\exists x, x'$ with $(f_1(x), f_0(x)) \leq (u, t)$ and $(f_1(x'), f_0(x')) \leq (u', t')$.

Geometric Proof of Strong Duality

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0\end{array}$$



Fact

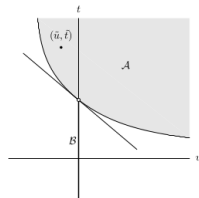
When f_0 and f_1 are convex, $\mathcal{A}$ is convex.

Proof

- Assume (u, t) and (u', t') are in $\mathcal{A}$
- $\exists x, x'$ with $(f_1(x), f_0(x)) \leq (u, t)$ and $(f_1(x'), f_0(x')) \leq (u', t')$.
- By Jensen's inequality
 $(f_1(\theta x + (1-\theta)x'), f_0(\theta x + (1-\theta)x')) \preceq (\theta u + (1-\theta)u', \theta t + (1-\theta)t')$

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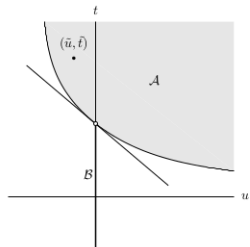
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- Therefore, segment connecting (u, t) and (u', t') also in $\mathcal{A}$.

Geometric Proof of Strong Duality

minimize $f_0(x)$
subject to $f_1(x) \leq 0$

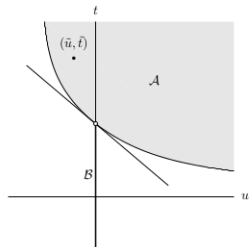


Theorem (Informal)

There is a choice of λ so that $g(\lambda) = p^*$. Therefore, strong duality holds.

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Theorem (Informal)

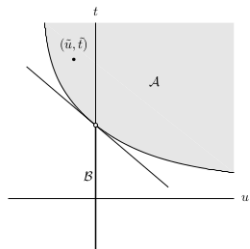
There is a choice of λ so that $g(\lambda) = p^*$. Therefore, strong duality holds.

Proof

- Recall $(0, p^*)$ is on the boundary of $\mathcal{A}$
- By the supporting hyperplane theorem, there is a supporting hyperplane to $\mathcal{A}$ at $(0, p^*)$
- Direction of the supporting hyperplane gives us an appropriate λ

I Lied (A little)

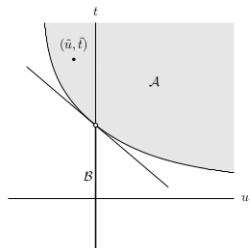
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- In our proof, we ignored a technicality that can prevent strong duality from holding.

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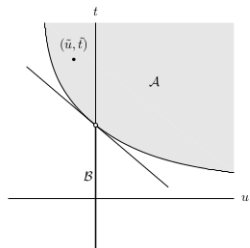
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- In our proof, we ignored a technicality that can prevent strong duality from holding.
- What if our supporting hyperplane H at $(0, p^*)$ is **vertical**?
 - The normal to H is perpendicular to the t axis
- In this case, no finite λ exists such that $(\lambda, 1)$ is normal to H .

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- In our proof, we ignored a technicality that can prevent strong duality from holding.
- What if our supporting hyperplane H at $(0, p^*)$ is **vertical**?
 - The normal to H is perpendicular to the t axis
- In this case, no finite λ exists such that $(\lambda, 1)$ is normal to H .
- Somewhat counterintuitively, this can happen even in simple convex optimization problems (though its somewhat rare in practice)

Violation of Strong Duality

$$\begin{array}{ll}\text{minimize} & e^{-x} \\ \text{subject to} & \frac{x^2}{y} \leq 0\end{array}$$

- Let domain be the region $y \geq 1$
- Problem is convex, with feasible region given by $x = 0$
- Optimal value is 1, at $x = 0$ and $y = 1$

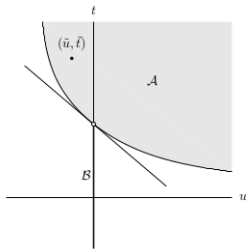
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- Optimal value is 1, at $x = 0$ and $y = 1$
- $\mathcal{A} = \mathbb{R}_{>0}^2 \cup (\{0\} \times [1, \infty])$
- Therefore, any supporting hyperplane to $\mathcal{A}$ at $(0, 1)$ must be vertical.
- Optimal dual value is 0; a duality gap of 1.

Slater's Condition

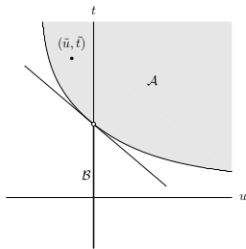
There exists a point $x \in \mathcal{D}$ where all inequality constraints are strictly satisfied (i.e. $f_i(x) < 0$). I.e. the optimization problem is **strictly feasible**.



- A sufficient condition for strong duality.
- Forces supporting hyperplane to be non-vertical

Slater's Condition

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- A sufficient condition for strong duality.
- Forces supporting hyperplane to be non-vertical
- Can be weakened to requiring strict feasibility only of non-affine constraints

Outline

1 The Lagrange Dual Problem

2 Duality

3 Optimality Conditions

Recall: Lagrangian Duality

Primal Problem

$$\min f_0(x)$$

s.t.

$$f_i(x) \leq 0, \quad \forall i = 1, \dots, m.$$

$$h_i(x) = 0, \quad \forall i = 1, \dots, k.$$

Dual Problem

$$\max g(\lambda, \nu)$$

s.t.

$$\lambda \succeq 0$$

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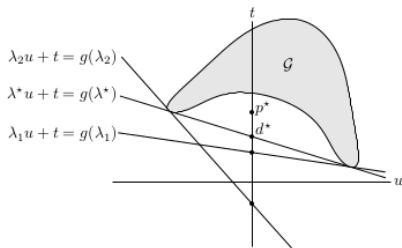
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Weak Duality

$$OPT(dual) \leq OPT(primal).$$



Recall: Lagrangian Duality

Primal Problem

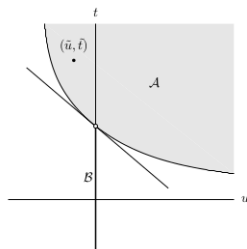
$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & \\ & f_i(x) \leq 0, \quad \forall i = 1, \dots, m. \\ & h_i(x) = 0, \quad \forall i = 1, \dots, k.\end{array}$$

Strong Duality

$$OPT(dual) = OPT(primal).$$

Dual Problem

$$\begin{array}{ll}\max & g(\lambda, \nu) \\ \text{s.t.} & \\ & \lambda \succeq 0\end{array}$$



Dual Solution as a Certificate

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- Dual solutions serves as a **certificate** of optimality
- If $f_0(x) = g(\lambda, \nu)$, and both are feasible, then both are optimal.

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 - $\text{OPT}(\text{primal})$ and $\text{OPT}(\text{dual})$ lie in the interval $[g(\lambda, \nu), f_0(x)]$

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 - $\text{OPT}(\text{primal})$ and $\text{OPT}(\text{dual})$ lie in the interval $[g(\lambda, \nu), f_0(x)]$
- **Primal-dual** algorithms use dual certificates to recognize optimality, or bound sub-optimality.

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Implications of Strong Duality

Primal Problem

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Facts

If strong duality holds, and x^* and (λ^*, ν^*) are feasible & optimal, then

- x^* minimizes $L(x, \lambda^*, \nu^*)$ over all x .
- $\lambda_i^* f_i(x^*) = 0$ for all $i = 1, \dots, m$. (Complementary Slackness)

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Proof

$$\begin{aligned}f_0(x^*) &= g(\lambda^*, \nu^*) = \min_x L(x, \lambda^*, \nu^*) \\ &\leq L(x^*, \lambda^*, \nu^*) = f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^k \nu_i^* h_i(x^*) \\ &\leq f_0(x^*)\end{aligned}$$

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Interpretation

- Lagrange multipliers (λ^*, ν^*) “simulate” the primal feasibility constraints
- Interpreting λ_i as the “value” of the i ’th constraint, at optimality only the binding constraints are “valuable”
 - Recall economic interpretation of LP

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KKT Conditions

Suppose the primal problem is convex and defined on an open domain, and moreover the constraint functions are differentiable everywhere in the domain. If strong duality holds, then x^* and (λ^*, ν^*) are optimal iff:

- x^* and (λ^*, ν^*) are feasible
- $\lambda_i^* f_i(x^*) = 0$ for all i (Complementary Slackness)
- $\nabla_x L(x^*, \lambda^*, \nu^*) = \nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^k \nu_i^* \nabla h_i(x^*) = 0$

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Why are KKT Conditions Useful?

- Derive an analytical solution to some convex optimization problems
- Gain structural insights

Example: Equality-constrained Quadratic Program

$$\begin{array}{ll}\text{minimize} & \frac{1}{2}x^\top Px + q^\top x + r \\ \text{subject to} & Ax = b\end{array}$$

- KKT Conditions: $Ax^* = b$ and $Px^* + q + A^\top \nu^* = 0$
- Simply a solution of a linear system with variables x^* and ν^* .
 - $m + n$ constraints and $m + n$ variables

Example: Market Equilibria (Fisher's Model)

- Buyers B , and goods G .
- Buyer i has utility u_{ij} for each unit of good G .
- Buyer i has budget m_i , and there's one divisible unit of each good.

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 - Prices p_j on items, such that each player can buy his favorite bundle that he can afford and the market clears (supply = demand).

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Eisenberg-Gale Convex Program

$$\begin{array}{ll} \text{maximize} & \sum_i m_i \log \sum_j u_{ij} x_{ij} \\ \text{subject to} & \sum_i x_{ij} \leq 1, \quad \text{for } j \in G. \\ & x \succeq 0 \end{array}$$

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Using KKT conditions, we can prove that the dual variables corresponding to the item supply constraints are market-clearing prices!