

CS675: Convex and Combinatorial Optimization
Fall 2026
Convex Functions

Instructor: Shaddin Dughmi

Outline

- 1 Convex Functions
- 2 Examples of Convex and Concave Functions
- 3 Convexity-Preserving Operations

Convex Functions

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** if the line segment between any points on the graph of f lies above f . i.e. if $x, y \in \mathbb{R}^n$ and $\theta \in [0, 1]$, then

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$



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Convex Functions

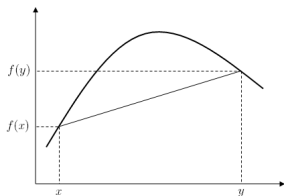
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- Analogous definition when domain of f is a convex subset of $\mathbb{R}^n$

Concave and Affine Functions

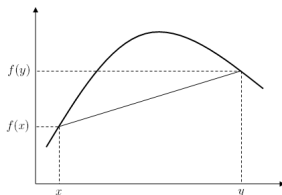


A function is $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **concave** if $-f$ is convex. Equivalently:

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$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **affine** if it is both concave and convex. Equivalently:

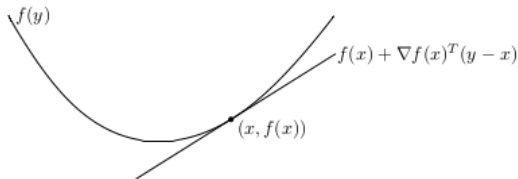
- Line segment between any points on the graph of f lies on the graph of f .
- $f(x) = a^\top x + b$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

We will now look at some equivalent definitions of convex functions

First Order Definition

A differentiable $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if the first-order approximation centered at any point x underestimates f everywhere.

$$f(y) \geq f(x) + (\nabla f(x))^T(y - x)$$

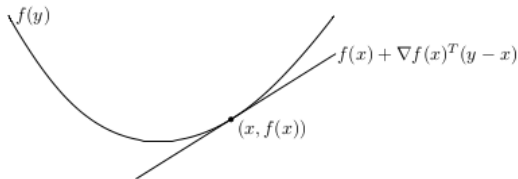


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- Local information $\rightarrow$ global information
- If $\nabla f(x) = 0$ then x is a global minimizer of f

Second Order Definition

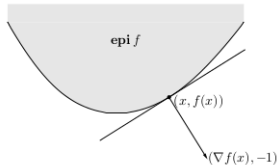
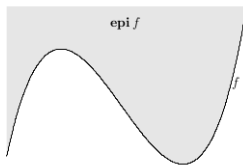
A twice differentiable $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if its Hessian matrix $\nabla^2 f(x)$ is positive semi-definite for all x . (We write $\nabla^2 f(x) \succeq 0$)

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Interpretation

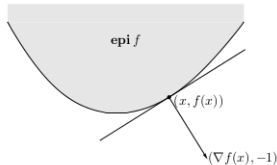
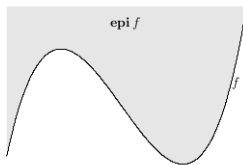
- Recall definition of PSD: $z^\top \nabla^2 f(x) z \geq 0$ for all $z \in \mathbb{R}^n$
- When $n = 1$, this is $f''(x) \geq 0$.
- More generally, $\frac{z^\top \nabla^2 f(x) z}{\|z\|^2}$ is the second derivative of f along the line $\{x + tz : t \in \mathbb{R}\}$. So if $\nabla^2 f(x) \succeq 0$ then f curves upwards along any line.
- Moving from x to $x + \delta \vec{z}$, infinitesimal change in gradient is $\delta \nabla^2 f(x) z$. When $\nabla^2 f(x) \succeq 0$, this is in roughly the same direction as $\vec{z}$.



Epigraph

The epigraph of f is the set of points above the graph of f . Formally,

$$\text{epi}(f) = \{(x, t) : t \geq f(x)\}$$



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Epigraph Definition

f is a convex function if and only if its epigraph is a convex set.

Jensen's Inequality (General Form)

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if

- For every $x_1, \dots, x_k$ in the domain of f , and $\theta_1, \dots, \theta_k \geq 0$ such that $\sum_i \theta_i = 1$, we have

$$f\left(\sum_i \theta_i x_i\right) \leq \sum_i \theta_i f(x_i)$$

- Given a probability measure $\mathcal{D}$ on the domain of f , and $x \sim \mathcal{D}$,

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Adding noise to x can only increase $f(x)$ in expectation.

Local and Global Optimality

Local minimum

x is a **local minimum** of f if there is an open ball B containing x where $f(y) \geq f(x)$ for all $y \in B$.

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When f is convex, x is a local minimum of f if and only if it is a global minimum.

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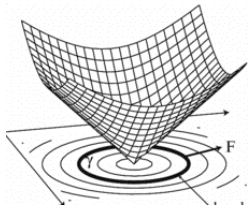
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- This fact underlies much of the tractability of convex optimization.

Sub-level sets

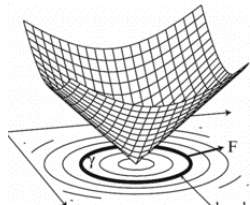


Level sets of $f(x, y) = \sqrt{x^2 + y^2}$

Sublevel set

The α -sublevel set of f is $\{x \in \text{domain}(f) : f(x) \leq \alpha\}$.

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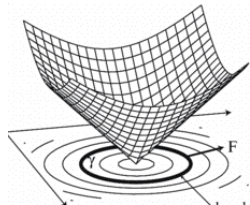
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Fact

Every sub-level set of a convex function is a convex set.

- This fact also underlies tractability of convex optimization

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Note: converse false, but nevertheless useful check.

Continuity

Real-valued convex functions are continuous on the interior of their domain.

Other Basic Properties

Continuity

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Extended-value extension

If a function $f : D \rightarrow \mathbb{R}$ is convex on its domain, and D is convex, then it can be extended to a convex function on $\mathbb{R}^n$ by setting $f(x) = \infty$ whenever $x \notin D$.

This simplifies notation. Resulting function $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \infty$ is “convex” with respect to the ordering on $\mathbb{R} \cup \infty$

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Functions on the reals

- Affine: $ax + b$
- Exponential: e^{ax} convex for any $a \in \mathbb{R}$
- Powers: x^a convex on $\mathbb{R}_{>0}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$
- Logarithm: $\log x$ concave on $\mathbb{R}_{>0}$.

Norms

Norms are convex.

$$\|\theta x + (1 - \theta)y\| \leq \|\theta x\| + \|(1 - \theta)y\| = \theta\|x\| + (1 - \theta)\|y\|$$

- Uses both norm axioms: triangle inequality, and homogeneity.
- Applies to matrix norms, such as the spectral norm (radius of induced ellipsoid)

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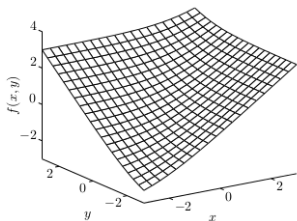
Max

$\max_i x_i$ is convex

$$\begin{aligned}\max_i (\theta x + (1 - \theta)y)_i &= \max_i (\theta x_i + (1 - \theta)y_i) \\ &\leq \max_i \theta x_i + \max_i (1 - \theta)y_i \\ &= \theta \max_i x_i + (1 - \theta) \max_i y_i\end{aligned}$$

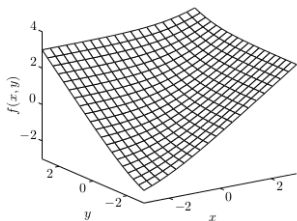
If I'm allowed to pick the maximum entry of θx and θy independently, I can do only better.

- Log-sum-exp: $\log(e^{x_1} + e^{x_2} + \dots + e^{x_n})$ is convex
- Geometric mean: $(\prod_{i=1}^n x_i)^{\frac{1}{n}}$ is concave
- Log-determinant: $\log \det X$ is concave
- Quadratic form: $x^\top A x$ is convex iff $A \succeq 0$
- Other examples in book



$$f(x, y) = \log(e^x + e^y)$$

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Proving convexity often comes down to case-by-case reasoning, involving:

- Definition: restrict to line and check Jensen's inequality
- Write down the Hessian and prove PSD
- Express as a combination of other convex functions through convexity-preserving operations (Next)

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Nonnegative Weighted Combinations

If $f_1, f_2, \dots, f_k$ are convex, and $w_1, w_2, \dots, w_k \geq 0$, then $g = w_1 f_1 + w_2 f_2 \dots + w_k f_k$ is convex.

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proof ($k = 2$)

$$\begin{aligned} g\left(\frac{x+y}{2}\right) &= w_1 f_1\left(\frac{x+y}{2}\right) + w_2 f_2\left(\frac{x+y}{2}\right) \\ &\leq w_1 \frac{f_1(x) + f_1(y)}{2} + w_2 \frac{f_2(x) + f_2(y)}{2} \\ &= \frac{g(x) + g(y)}{2} \end{aligned}$$

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Extends to integrals $g(x) = \int_y w(y) f_y(x)$ with $w(y) \geq 0$, and therefore expectations $\mathbf{E}_y f_y(x)$.

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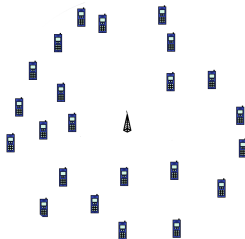
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Worth Noting

Minimizing the expectation of a random convex cost function is also a convex optimization problem!

- A **stochastic** convex optimization problem is a convex optimization problem.

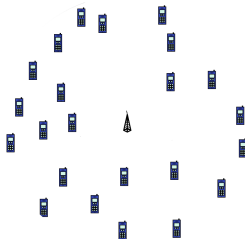
Example: Stochastic Facility Location



Average Distance

- k customers located at $y_1, y_2, \dots, y_k \in \mathbb{R}^n$
- If I place a facility at $x \in \mathbb{R}^n$, average distance to a customer is $g(x) = \sum_i \frac{1}{k} \|x - y_i\|$

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- If I place a facility at $x \in \mathbb{R}^n$, average distance to a customer is $g(x) = \sum_i \frac{1}{k} \|x - y_i\|$
- Since distance to any one customer is convex in x , so is the average distance.
- Extends to probability measure over customers

Implication

Convex functions are a convex cone in the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}$.

The set of convex functions is the intersection of an infinite set of homogeneous linear inequalities indexed by x, y, θ

$$f(\theta x + (1 - \theta)y) - \theta f(x) - (1 - \theta)f(y) \leq 0$$

Composition with Affine Function

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, and $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$, then

$$g(x) = f(Ax + b)$$

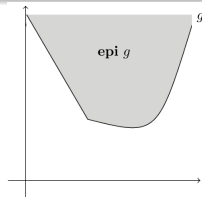
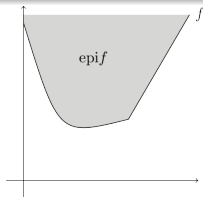
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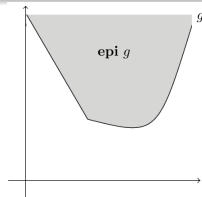
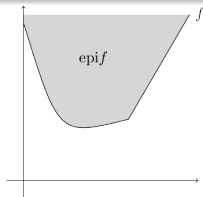
$$(x, t) \in \mathbf{graph}(g) \iff t = g(x) = f(Ax+b) \iff (Ax+b, t) \in \mathbf{graph}(f)$$

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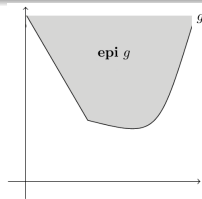
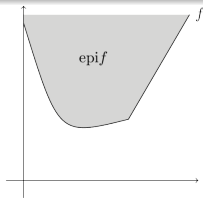
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$$(x, t) \in \mathbf{graph}(g) \iff t = g(x) = f(Ax+b) \iff (Ax+b, t) \in \mathbf{graph}(f)$$

$$(x, t) \in \mathbf{epi}(g) \iff t \geq g(x) = f(Ax+b) \iff (Ax+b, t) \in \mathbf{epi}(f)$$

$\mathbf{epi}(g)$ is the inverse image of $\mathbf{epi}(f)$ under the affine mapping
 $(x, t) \rightarrow (Ax + b, t)$

Examples

- $\|Ax + b\|$ is convex
- $\max(Ax + b)$ is convex
- $\log(e^{a_1^\top x + b_1} + e^{a_2^\top x + b_2} + \dots + e^{a_n^\top x + b_n})$ is convex

Maximum

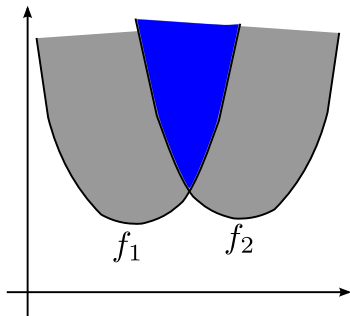
If f_1, f_2 are convex, then $g(x) = \max \{f_1(x), f_2(x)\}$ is also convex.

Generalizes to the maximum of any number of functions, $\max_{i=1}^k f_i(x)$, and also to the supremum of an infinite set of functions $\sup_y f_y(x)$.

Maximum

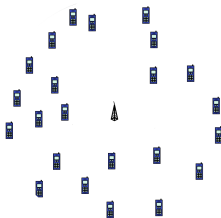
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$$\text{epi } g = \text{epi } f_1 \cap \text{epi } f_2$$

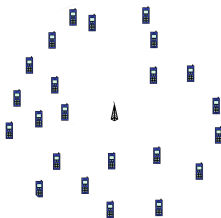
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Maximum Distance

- k customers located at $y_1, y_2, \dots, y_k \in \mathbb{R}^n$
- If I place a facility at $x \in \mathbb{R}^n$, maximum distance to a customer is $g(x) = \max_i ||x - y_i||$

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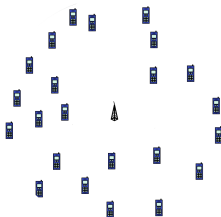


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Since distance to any one customer is convex in x , so is the worst-case distance.

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Worth Noting

When a convex cost function is uncertain, minimizing the **worst-case** cost is also a convex optimization problem!

- A **robust** (in the worst-case sense) convex optimization problem is a convex optimization problem.

Other Examples

- Maximum eigenvalue of a symmetric matrix A is convex in A

$$\max \{v^T A v : \|v\| = 1\}$$

- Sum of k largest components of a vector x is convex in x

$$\max \left\{ \vec{1}_S \cdot x : |S| = k \right\}$$

Minimization

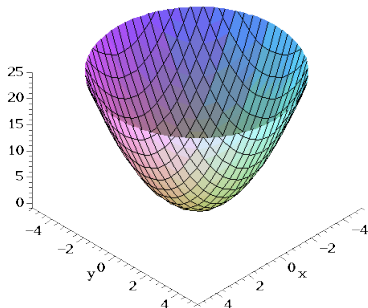
If $f(x, y)$ is convex and $\mathcal{C}$ is convex and nonempty, then $g(x) = \inf_{y \in \mathcal{C}} f(x, y)$ is convex.

Minimization

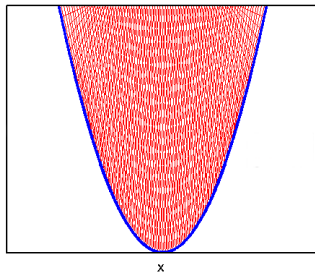
If $f(x, y)$ is convex and $\mathcal{C}$ is convex and nonempty, then $g(x) = \inf_{y \in \mathcal{C}} f(x, y)$ is convex.

Proof (for $\mathcal{C} = \mathbb{R}^k$)

$\text{epi } g$ is the projection of $\text{epi } f$ onto hyperplane $y = 0$.



$$f(x, y) = x^2 + y^2$$



$$g(x) = x^2$$

Example

Distance from a convex set $\mathcal{C}$

$$f(x) = \inf_{y \in \mathcal{C}} \|x - y\|$$

Composition Rules

If $g : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $h : \mathbb{R}^k \rightarrow \mathbb{R}$, then $f = h \circ g$ is convex if

- g_i are convex, and h is convex and nondecreasing in each argument.
- g_i are concave, and h is convex and nonincreasing in each argument.

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Proof ($n = k = 1$, twice differentiable)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

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Proof of first case

$$\begin{aligned} g(\theta x + (1 - \theta)y) &\preceq \theta g(x) + (1 - \theta)g(y) && \text{(component-wise)} \\ h(g(\theta x + (1 - \theta)y)) &\leq h(\theta g(x) + (1 - \theta)g(y)) && (h \text{ non-decreasing}) \\ &\leq \theta h(g(x)) + (1 - \theta)h(g(y)) && (h \text{ convex}) \end{aligned}$$

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Proof of second case is almost identical

Perspective

If f is convex then $g(x, t) = tf(x/t)$ is also convex over $t > 0$.

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$\mathbf{epi} g$ is inverse image of $\mathbf{epi} f$ under the perspective function
 $(x, t, y) \rightarrow (x/t, y/t)$.