

CS675: Convex and Combinatorial Optimization
Fall 2026
Convex Optimization Problems

Instructor: Shaddin Dughmi

Outline

- 1 Convex Optimization Basics
- 2 Common Classes
- 3 Interlude: Positive Semi-Definite Matrices
- 4 More Convex Optimization Problems

Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X}\end{array}$$

- $\mathcal{X} \subseteq \mathbb{R}^n$ is convex, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex
- Terminology: decision variable(s), objective function, feasible set, optimal solution/value, ϵ -optimal solution/value

Standard Form

Instances typically formulated in the following **standard form**

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad \text{for } i \in \mathcal{C}_1. \\ & \langle a_i, x \rangle = b_i, \quad \text{for } i \in \mathcal{C}_2.\end{array}$$

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 - Recall: every convex set is the intersection of halfspaces

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- In principle, every convex optimization problem can be formulated in this form (possibly implicitly)
 - Recall: every convex set is the intersection of halfspaces
- When there is no objective function (or, equivalently, $f(x) = 0$ for all x), we say this is **convex feasibility problem**

Local and Global Optimality

$x \in \mathcal{X}$ is **locally optimal** if $\exists$ open ball B centered at x s.t. $f(x) \leq f(y)$ for all $y \in B \cap \mathcal{X}$. It is **globally optimal** if it's an optimal solution.

Fact

For a convex optimization problem, every locally optimal feasible solution is globally optimal.

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- Let x be locally optimal, and y be any other feasible point.
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- By local optimality $f(x) \leq f(\theta x + (1 - \theta)y)$ for θ sufficiently close to 1.

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- The line segment from x to y is contained in the feasible set.
- By local optimality $f(x) \leq f(\theta x + (1 - \theta)y)$ for θ sufficiently close to 1.
- Jensen's inequality then implies that y is suboptimal.

$$f(x) \leq f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

$$f(x) \leq f(y)$$

Representation

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Explicit Representation

A family of linear programs of the following form

$$\begin{array}{ll}\text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0\end{array}$$

may be described by $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

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Oracle Representation

At their most abstract, convex optimization problems of the following form

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X}\end{array}$$

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Given additional data about instances of the problem, namely a range $[L, H]$ for its optimal value and a ball of volume V containing $\mathcal{X}$, the ellipsoid method returns an ϵ -optimal solution using only $\text{poly}(n, \log(\frac{H-L}{\epsilon}), \log V)$ oracle calls.

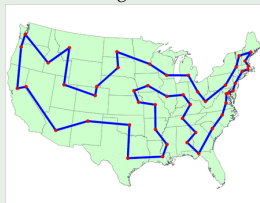
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In Between

Consider the following **fractional relaxation** of the Traveling Salesman Problem, described by a network (V, E) and distances d_e on $e \in E$.

$$\begin{array}{ll}\min & \sum_e d_e x_e \\ \text{s.t.} & \\ & \sum_{e \in \delta(S)} x_e \geq 2, \quad \forall S \subset V, S \neq \emptyset. \\ & x \succeq 0\end{array}$$



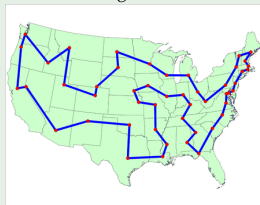
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Representation of LP is implicit, in the form of a network. Using this representation, separation oracles can be implemented efficiently, and used as subroutines in the ellipsoid method.

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- Next up: we look at some common classes of convex optimization problems
- Technically, not all of them will be convex in their natural representation
- However, we will show that they are “equivalent” to a convex optimization problem

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Note

Deciding whether an optimization problem is equivalent to a tractable convex optimization problem is, in general, a black art honed by experience. There is no silver bullet.

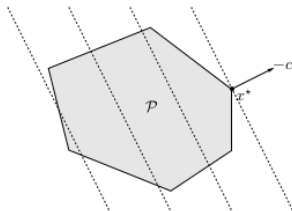
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Linear Programming

We have already seen linear programming

$$\begin{array}{ll}\text{minimize} & \langle c, x \rangle \\ \text{subject to} & Ax \leq b\end{array}$$



Linear Fractional Programming

Generalizes linear programming

$$\begin{array}{ll}\text{minimize} & \frac{\langle c, x \rangle + d}{\langle e, x \rangle + f} \\ \text{subject to} & Ax \preceq b \\ & \langle e, x \rangle + f > 0\end{array}$$

- The objective is quasiconvex (in fact, quasilinear) over the open halfspace where the denominator is positive.

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Example: Optimal Production Variant

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j dollars per unit, and requires an investment of e_j dollars per unit to produce, with f as a fixed cost
- Facility wants to maximize “Return rate on investment”

$$\begin{array}{ll}\text{maximize} & \frac{\langle c, x \rangle}{\langle e, x \rangle + f} \\ \text{subject to} & \langle a_i, x \rangle \leq b_i, \quad \text{for } i = 1, \dots, m. \\ & x_j \geq 0, \quad \text{for } j = 1, \dots, n.\end{array}$$

Definition

- A **monomial** is a function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ of the form

$$f(x) = cx_1^{a_1}x_2^{a_2}\dots x_n^{a_n},$$

where $c \geq 0$, $a_i \in \mathbb{R}$.

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Interpretation

GP model volume/area minimization problems, subject to constraints.

Example: Designing a Suitcase

- A suitcase manufacturer is designing a suitcase
- Variables: h, w, d
- Want to minimize surface area $2(hw + hd + wd)$ (i.e. amount of material used)
- Have a target volume $hwd \geq 5$
- Practical/aesthetic constraints limit aspect ratio: $h/w \leq 2, h/d \leq 3$
- Constrained by airline to $h + w + d \leq 7$

$$\begin{array}{ll}\text{minimize} & 2hw + 2hd + 2wd \\ \text{subject to} & h^{-1}w^{-1}d^{-1} \leq \frac{1}{5} \\ & hw^{-1} \leq 2 \\ & hd^{-1} \leq 3 \\ & h + w + d \leq 7 \\ & h, w, d \geq 0\end{array}$$

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More interesting applications involve optimal component layout in chip design.

Designing a Suitcase in Convex Form

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Change of variables to $\tilde{h} = \log h$, $\tilde{w} = \log w$, $\tilde{d} = \log d$

$$\begin{array}{ll}\text{minimize} & 2e^{\tilde{h}+\tilde{w}} + 2e^{\tilde{h}+\tilde{d}} + 2e^{\tilde{w}+\tilde{d}} \\ \text{subject to} & e^{-\tilde{h}-\tilde{w}-\tilde{d}} \leq \frac{1}{5} \\ & e^{\tilde{h}-\tilde{w}} \leq 2 \\ & e^{\tilde{h}-\tilde{d}} \leq 3 \\ & e^{\tilde{h}} + e^{\tilde{w}} + e^{\tilde{d}} \leq 7\end{array}$$

Geometric Programs in Convex Form

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where f_i 's are posynomials, h_i 's are monomials, and $b_i > 0$ (wlog 1).

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- In their natural parametrization by $x_1, \dots, x_n \in \mathbb{R}_+$, geometric programs are not convex optimization problems
- However, the feasible set and objective function are convex in the variables $y_1, \dots, y_n \in \mathbb{R}$ where $y_i = \log x_i$

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- Each monomial $cx_1^{a_1}x_2^{a_2}\dots x_k^{a_k}$ can be rewritten as a convex function $ce^{a_1y_1+a_2y_2+\dots+a_ky_k}$
- Therefore, each posynomial becomes the sum of these convex exponential functions
- Inequality constraints and objective become convex
- Equality constraint $cx_1^{a_1}x_2^{a_2}\dots x_k^{a_k} = b$ reduces to an affine constraint $a_1y_1 + a_2y_2 \dots a_ky_k = \log \frac{b}{c}$

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Symmetric Matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is **symmetric** if and only if it is square and $A_{ij} = A_{ji}$ for all i, j .

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Fact

A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if and only if it is **orthogonally diagonalizable**.

- i.e. $A = QDQ^T$ where Q is an **orthogonal matrix** and $D = \text{diag}(\lambda_1, \dots, \lambda_n)$.
- The columns of Q are the (normalized) eigenvectors of A , with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$
- Equivalently: As a linear operator, A scales the space along an orthonormal basis Q
- The scaling factor λ_i along direction q_i may be negative, positive, or 0.

Positive Semi-Definite Matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is **positive semi-definite** if it is symmetric and moreover all its eigenvalues are nonnegative.

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Note

Positive definite, negative semi-definite, and negative definite defined similarly.

Geometric Intuition for PSD Matrices



- For $A \succeq 0$, let $q_1, \dots, q_n$ be the orthonormal eigenbasis for A , and let $\lambda_1, \dots, \lambda_n \geq 0$ be the corresponding eigenvalues.
- The linear operator $x \rightarrow Ax$ scales the q_i component of x by λ_i
- When applied to every x in the unit ball, the image of A is an ellipsoid centered at the origin with **principal directions** $q_1, \dots, q_n$ and corresponding diameters $2\lambda_1, \dots, 2\lambda_n$
 - When A is **positive definite** (i.e. $\lambda_i > 0$), and therefore invertible, the ellipsoid is the set $\{y : y^T (AA^T)^{-1} y \leq 1\}$

Useful Properties of PSD Matrices

If $A \succeq 0$, then

- $x^T A x \geq 0$ for all x
- A has a positive semi-definite square root $A^{\frac{1}{2}}$
 - $A^{\frac{1}{2}} = Q \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) Q^T$
- $A = B^T B$ for some matrix B .
 - Interpretation: PSD matrices encode the “pairwise similarity” relationships of a family of vectors. A_{ij} is dot product of the i th and j th columns of B .
 - Interpretation: The quadratic form $x^T A x$ is the length of a linear transformation of x , namely $\|Bx\|_2^2$
- The quadratic function $x^T A x$ is convex
- A can be expressed as a sum of vector outer-products
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As it turns out, each of the above is also sufficient for $A \succeq 0$ (assuming A is symmetric).

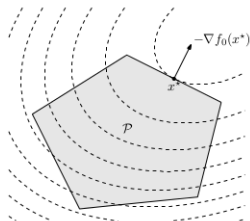
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- 2 Common Classes
- 3 Interlude: Positive Semi-Definite Matrices
- 4 More Convex Optimization Problems

Quadratic Programming

Minimizing convex quadratic fn over a polyhedron. Require $P \succeq 0$.

$$\begin{array}{ll}\text{minimize} & x^\top P x + \langle c, x \rangle + d \\ \text{subject to} & A x \preceq b\end{array}$$

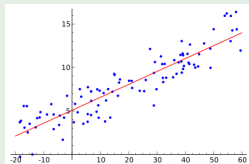


- When $P \succ 0$, objective can be rewritten as $(x - x_0)^\top P (x - x_0)$ for some center x_0 (might need to change d , which is immaterial)
 - Sublevel sets are scaled copies of an ellipsoid centered at x_0

Constrained Least Squares

Given a set of measurements $(a_1, b_1), \dots, (a_m, b_m)$, where $a_i \in \mathbb{R}^n$ is the i 'th input and $b_i \in \mathbb{R}$ is the i 'th output, fit a linear function minimizing mean square error, subject to known bounds on the linear coefficients.

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 = x^\top A^\top Ax - 2b^\top Ax + \langle b, b \rangle \\ \text{subject to} & l_i \leq x_i \leq u_i, \quad \text{for } i = 1, \dots, n. \end{array}$$



Distance Between Polyhedra

Given two polyhedra $Ax \preceq b$ and $Cx \preceq d$, find the distance between them.

$$\begin{array}{ll} \text{minimize} & \|z\|_2^2 = z^\top I z \\ \text{subject to} & z = y - x \\ & Ax \preceq b \\ & By \preceq d \end{array}$$

Conic Optimization Problems

This is an umbrella term for problems of the following form

$$\begin{array}{ll}\text{minimize} & c^\top x \\ \text{subject to} & Ax + b \in K\end{array}$$

Where K is a convex cone (e.g. $\mathbb{R}_+^n$, positive semi-definite matrices, etc). Evidently, such optimization problems are convex.

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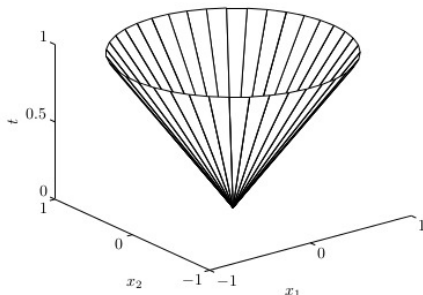
As shorthand, the cone containment constraint is often written using **generalized inequalities**

- $Ax + b \succeq_K 0$
- $-Ax \preceq_K b$
- ...

Example: Second Order Cone Programming

We will exhibit an example of a conic optimization problem with K as the **second order cone**

$$K = \{(x, t) : \|x\|_2 \leq t\}$$



Example: Second Order Cone Programming

Linear Program with Random Constraints

Consider the following optimization problem, where each a_i is a gaussian random variable with mean $\bar{a}_i$ and covariance matrix Σ_i .

minimize $c^\top x$

subject to $\langle a_i, x \rangle \leq b_i$ **w.p.** at least 0.9, for $i = 1, \dots, m$.

- $u_i := \langle a_i, x \rangle$ is a univariate normal r.v. with mean $\bar{u}_i := \bar{a}_i^\top x$ and stddev $\sigma_i := \sqrt{x^\top \Sigma_i x} = \|\Sigma_i^{\frac{1}{2}} x\|_2$

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- $u_i \leq b_i$ with probability $\phi(\frac{b_i - \bar{u}_i}{\sigma_i})$, where ϕ is the CDF of the standard normal random variable.
- Since we want this probability to exceed 0.9, we require that

$$\frac{b_i - \bar{u}_i}{\sigma_i} \geq \phi^{-1}(0.9) \approx 1.3 \approx 1/0.77$$

$$\|\Sigma_i^{\frac{1}{2}} x\|_2 \leq 0.77(b_i - \bar{a}_i^\top x)$$

Semi-Definite Programming

These are conic optimization problems where the cone in question is the set of positive semi-definite matrices.

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 \dots x_n F_n + G \succeq 0\end{array}$$

Where $F_1, \dots, F_n$ are matrices, and $\succeq$ refers to the positive semi-definite cone S_+^n .

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Examples

- Fitting a distribution, say a Gaussian, to observed data. Variable is a positive semi-definite covariance matrix.
- As a relaxation to combinatorial problems that encode pairwise relationships: e.g. finding the maximum cut of a graph.

Example: Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of V into $(S, V \setminus S)$ maximizing number of edges with exactly one end in S .

$$\begin{array}{ll} \text{maximize} & \sum_{(i,j) \in E} \frac{1-x_i x_j}{2} \\ \text{subject to} & x_i \in \{-1, 1\}, \quad \text{for } i \in V. \end{array}$$

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Vector Program relaxation

$$\begin{array}{ll}\text{maximize} & \sum_{(i,j) \in E} \frac{1 - x_i \cdot x_j}{2} \\ \text{subject to} & \|x_i\|_2 = 1, \quad \text{for } i \in V. \\ & x_i \in \mathbb{R}^n, \quad \text{for } i \in V.\end{array}$$

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SDP Relaxation

$$\begin{array}{ll}\text{maximize} & \sum_{(i,j) \in E} \frac{1-X_{ij}}{2} \\ \text{subject to} & X_{ii} = 1, \quad \text{for } i \in V. \\ & X \in S_+^n\end{array}$$