

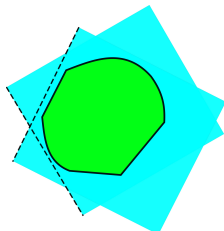
CS675: Convex and Combinatorial Optimization
Fall 2026
Geometric Duality of Convex Sets and Functions

Instructor: Shaddin Dughmi

Outline

- 1 Convexity and Duality
- 2 Duality of Convex Sets
- 3 Duality of Convex Functions

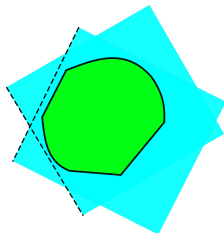
Duality Correspondences



There are two equivalent ways to represent a convex set

- The family of points in the set (standard or “primal” representation)
- The set of halfspaces containing the set (“dual” representation)

Duality Correspondances

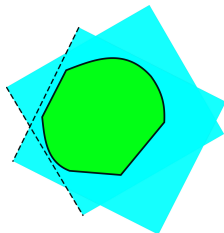


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Duality Correspondences



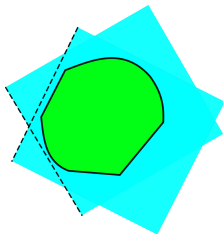
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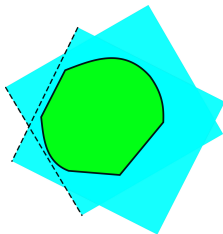
Definition

“**Duality**” is a woefully overloaded mathematical term for a relation that groups elements of a set into “dual” pairs.



Theorem

A closed convex set S is the intersection of all closed halfspaces $\mathcal{H}$ containing it.

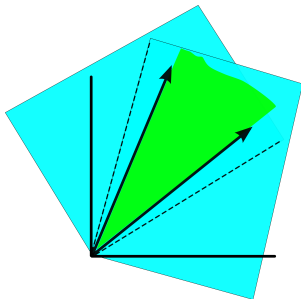


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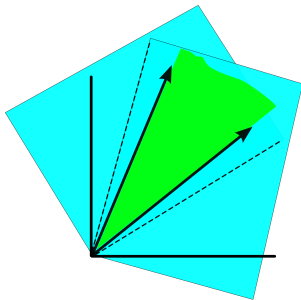
Proof

- Clearly, $S \subseteq \bigcap_{H \in \mathcal{H}} H$
- To prove equality, consider $x \notin S$
- By the separating hyperplane theorem, there is a hyperplane separating S from x
- Therefore there is $H \in \mathcal{H}$ with $x \notin H$, hence $x \notin \bigcap_{H \in \mathcal{H}} H$



Theorem

A closed convex cone K is the intersection of all closed homogeneous halfspaces $\mathcal{H}$ containing it.

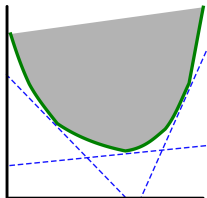


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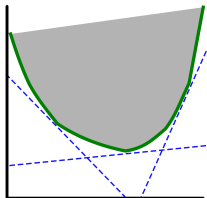
Proof

- For every non-homogeneous halfspace $\langle a, x \rangle \leq b$ containing K , the smaller homogeneous halfspace $\langle a, x \rangle \leq 0$ contains K as well.
- Therefore, can discard non-homogeneous halfspaces without changing the intersection



Theorem

A convex function is the point-wise supremum of all affine functions under-estimating it everywhere.



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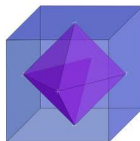
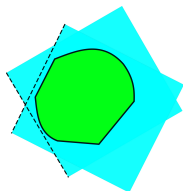
Proof

- $\text{epi } f$ convex, therefore is the intersection of family of halfspaces $\mathcal{H}$
- Each $h \in \mathcal{H}$ can be written as $\langle a, x \rangle - t \leq b$, for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. (Why?)
 - Constrains $(x, t) \in \text{epi } f$ to $\langle a, x \rangle - b \leq t$
- $f(x)$ is the lowest t s.t. $(x, t) \in \text{epi } f$
- Therefore, $f(x)$ is the point-wise maximum of $\langle a, x \rangle - b$ over all halfspaces $h(a, b) \in \mathcal{H}$.

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Polar Duality of Convex Sets



One way of representing all the halfspaces containing a convex set.

Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The **polar** of S is defined as follows:

$$S^\circ = \{y : \langle y, x \rangle \leq 1 \text{ for all } x \in S\}$$

Note

- Every halfspace $\langle a, x \rangle \leq b$ with $b \neq 0$ can be written as a “normalized” inequality $\langle y, x \rangle \leq 1$, by dividing by b .
- S° can be thought of as the normalized representations of halfspaces containing S .

$$S^\circ = \{y : \langle y, x \rangle \leq 1 \text{ for all } x \in S\}$$

Properties of the Polar

- 1 $S^{\circ\circ} = S$
- 2 S° is a closed convex set containing the origin
- 3 When 0 is in the interior of S , then S° is bounded.

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Properties of the Polar

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- 2 Follows from representation as intersection of halfspaces
 - 3 S contains an ϵ -ball centered at the origin, so S° is contained in the $\frac{1}{\epsilon}$ ball centered at the origin.
 - Take $y \in S^\circ$
 - $x := \epsilon \frac{y}{\|y\|_2} \in S$
 - $1 \geq \langle y, x \rangle = \epsilon \|y\|_2$, so $\|y\|_2 \leq \frac{1}{\epsilon}$

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Properties of the Polar

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$$S^{\circ\circ} = \{x : \langle x, y \rangle \leq 1 \text{ for all } y \in S^\circ\}$$

- 1
 - $S \subseteq S^{\circ\circ}$ is easy: $\hat{x} \in S \implies \forall y \in S^\circ \langle \hat{x}, y \rangle \leq 1 \implies \hat{x} \in S^{\circ\circ}$
 - Take $\hat{x} \notin S$, by SSHT and $0 \in S$, there is a halfspace $\langle z, x \rangle \leq 1$ containing S but not $\hat{x}$ (i.e. $\langle z, \hat{x} \rangle > 1$)
 - $z \in S^\circ$, therefore $\hat{x} \notin S^{\circ\circ}$

$$S^\circ = \{y : \langle y, x \rangle \leq 1 \text{ for all } x \in S\}$$

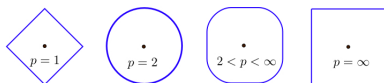
Properties of the Polar

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Note

When S is non-convex, $S^\circ = (\text{convexhull}(S))^\circ$, and $S^{\circ\circ} = \text{convexhull}(S)$.

Examples

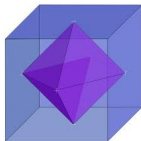


The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in $\mathbb{R}^2$.

Norm Balls

- The polar of the Euclidean unit ball is itself (we say it is **self-dual**)
- The polar of the 1-norm ball is the ∞ -norm ball
- More generally, the p -norm ball is dual to the q -norm ball, where $\frac{1}{p} + \frac{1}{q} = 1$

Examples

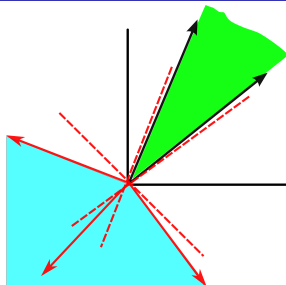


Polytopes

Given a polytope P represented as $Ax \preceq \vec{1}$, the polar P° is the convex hull of the rows of A .

- Facets of P correspond to vertices of P° .
- Dually, vertices of P correspond to facets of P° .

Polar Duality of Convex Cones



Polar duality takes a simplified form when applied to cones

Polar

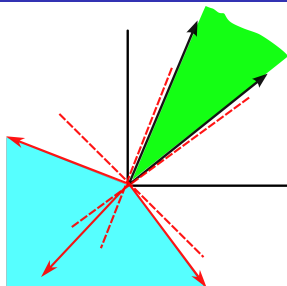
The polar of a closed convex cone K is given by

$$K^\circ = \{y : \langle y, x \rangle \leq 0 \text{ for all } x \in K\}$$

Note

- $\forall x \in K \langle y, x \rangle \leq 1 \iff \forall x \in K \langle y, x \rangle \leq 0$
- K° represents all homogeneous halfspaces containing K .

Polar Duality of Convex Cones



Polar duality takes a simplified form when applied to cones

Polar

The polar of a closed convex cone K is given by

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Dual Cone

By convention, $K^* = -K^\circ$ is referred to as the **dual cone** of K .

$$K^* = \{y : \langle y, x \rangle \geq 0 \text{ for all } x \in K\}$$

$$K^\circ = \{y : \langle y, x \rangle \leq 0 \text{ for all } x \in K\}$$

Properties of the Polar Cone

- 1 $K^{\circ\circ} = K$
- 2 K° is a closed convex cone

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Properties of the Polar Cone

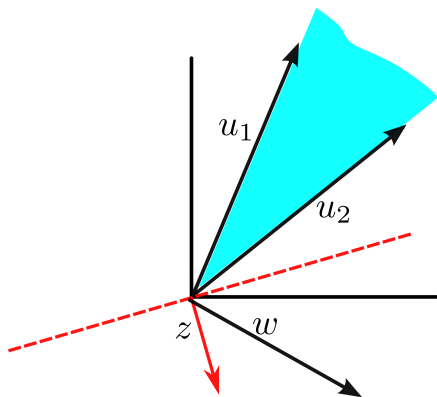
- 1 $K^{\circ\circ} = K$
 - 2 K° is a closed convex cone
-
- 1 Same as before
 - 2 Intersection of homogeneous halfspaces

Examples

- The polar of a subspace is its orthogonal complement
- The polar cone of the nonnegative orthant is the nonpositive orthant
 - Self-dual
- The polar of a polyhedral cone $Ax \preceq 0$ is the conic hull of the rows of A
- The polar of the cone of positive semi-definite matrices is the cone of negative semi-definite matrices
 - Self-dual

Recall: Farkas' Lemma

Let K be a **closed convex cone** and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $\langle z, x \rangle \leq 0$ for all $x \in K$, and $\langle z, w \rangle > 0$.

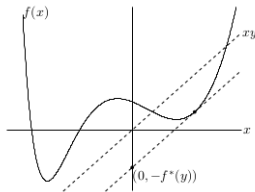


Equivalently: there is $z \in K^\circ$ with $\langle z, w \rangle > 0$.

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Conjugation of Convex Functions



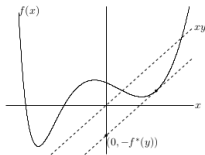
Conjugate

For a function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$, the **conjugate** of f is

$$f^*(y) = \sup_x (\langle y, x \rangle - f(x))$$

Note

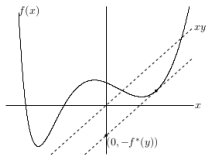
- $f^*(y)$ is the minimal value of β such that the affine function $\langle y, x \rangle - \beta$ underestimates $f(x)$ everywhere.
- Equivalently, the distance we need to lower the hyperplane $\langle y, x \rangle - t = 0$ in order to get a supporting hyperplane to $\text{epi } f$.
- $\langle y, x \rangle - t = f^*(y)$ are the supporting hyperplanes of $\text{epi } f$



$$f^*(y) = \sup_x (\langle y, x \rangle - f(x))$$

Properties of the Conjugate

- 1 $f^{**} = f$ when f is convex
- 2 f^* is a convex function
- 3 $xy \leq f(x) + f^*(y)$ for all $x, y \in \mathbb{R}^n$ (Fenchel's Inequality)

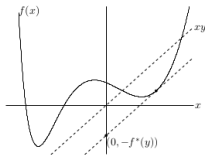


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- 2 Supremum of affine functions of y
- 3 By definition of $f^*(y)$

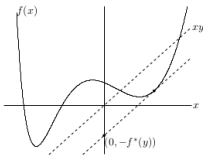


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- 1 • $f^{**}(x) = \sup_y \langle y, x \rangle - f^*(y)$ by definition

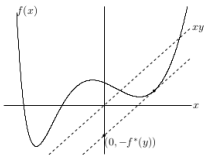


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 - For fixed y , $f^*(y)$ is minimal β such that $\langle y, x \rangle - \beta$ underestimates f .

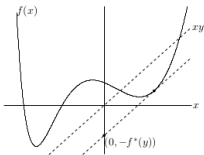


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 - For fixed y , $f^*(y)$ is minimal β such that $\langle y, x \rangle - \beta$ underestimates f .
 - Therefore $f^{**}(x)$ is the maximum, over all y , of affine underestimates $\langle y, x \rangle - \beta$ of f



$$f^*(y) = \sup_x (\langle y, x \rangle - f(x))$$

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 - Therefore $f^{**}(x)$ is the maximum, over all y , of affine underestimates $\langle y, x \rangle - \beta$ of f
 - By our earlier characterization, this is equal to f when f is convex.

Examples

- Affine function: $f(x) = ax + b$. Conjugate has $f^*(a) = -b$, and ∞ elsewhere
- $f(x) = x^2/2$ is self-conjugate
- Exponential: $f(x) = e^x$. Conjugate has $f^*(y) = y \log y - y$ for $y \in \mathbb{R}_+$, and ∞ elsewhere.
- Convex Quadratic: $f(x) = \frac{1}{2}x^\top Qx$ with Q positive definite. Conjugate is $f^*(y) = \frac{1}{2}y^\top Q^{-1}y$
- Log-sum-exp: $f(x) = \log(\sum_i e^{x_i})$. Conjugate has $f^*(y) = \sum_i y_i \log y_i$ for $y \succeq 0$ and $\langle \vec{1}, y \rangle = 1$, ∞ otherwise.